

Appendix E Wedge Equations

1. General Wedge Equation.

a. Nomenclature

α_i = angle between failure plane and horizontal

ϕ_i = internal friction angle of material in i th wedge. Earth and rock wedges only.

c_i = cohesive strength of material in i th wedge. Earth and rock wedges only.

W_i = weight of material in i th wedge.

V_i = surcharge load acting on i th wedge.

H_{Li} = horizontal force on i th wedge, acting to the right.

H_{Ri} = horizontal force on i th wedge, acting to left.

U_i = uplift (water) load acting normal to failure plane.

N_i = force acting normal to failure plane of i th wedge.

T_i = shear force acting parallel to failure plane of i th wedge.

P_i = horizontal force due to the i th wedge.

L_i = length of wedge base.

b. Sign Convention. The x and y axes are to the right and upward, respectively. The t and n axes are tangent and normal to the failure plane. Where the failure plane is not horizontal, the angle from the x axis to the t axis is determined by the right hand rule.

c. Equilibrium equations.

$$\Sigma F_n = 0$$

$$0 = N_i + U_i - W_i \cos \alpha_i - V_i \cos \alpha_i - H_{Li} \sin \alpha_i + H_{Ri} \sin \alpha_i - (P_{i-1} - P_i) \sin \alpha_i$$

$$N_i = (W_i + V_i) \cos \alpha_i - U_i + (H_{Li} - H_{Ri}) \sin \alpha_i + (P_{i-1} - P_i) \sin \alpha_i$$

$$\Sigma F_t = 0$$

$$0 = -T_i - W_i \sin \alpha_i - V_i \sin \alpha_i + H_{Li} \cos \alpha_i - H_{Ri} \cos \alpha_i + (P_{i-1} - P_i) \cos \alpha_i$$

$$T_i = (H_{Li} - H_{Ri}) \cos \alpha_i - (W_i + V_i) \sin \alpha_i + (P_{i-1} - P_i) \cos \alpha_i$$

d. Mohr-Coulomb failure criterion.

$$T_F = N_i \tan \phi_i + c_i L_i$$

e. Sliding factor of safety definition.

$$FS_i = \frac{T_F}{T_i} = \frac{N_i \tan \phi_i + c_i L_i}{T_i}$$

f. Governing wedge equation.

$$FS_i = \frac{[(W_i + V_i) \cos \alpha_i - U_i + (H_{Li} - H_{Ri}) \sin \alpha_i + (P_{i-1} - P_i) \sin \alpha_i] \tan \phi_i + c_i L_i}{(H_{Li} - H_{Ri}) \cos \alpha_i - (W_i + V_i) \sin \alpha_i + (P_{i-1} - P_i) \cos \alpha_i} \quad (\text{E-1})$$

$$(P_{i-1} - P_i) = \frac{[(W_i + V_i) \cos \alpha_i - U_i + (H_{Li} - H_{Ri}) \sin \alpha_i] \frac{\tan \phi_i}{FS_i} - (H_{Li} - H_{Ri}) \cos \alpha_i + (W_i + V_i) \sin \alpha_i + \frac{c_i L_i}{FS_i}}{\cos \alpha_i - \frac{\tan \phi_i \sin \alpha_i}{FS_i}}$$

A negative value of the difference $(P_{i-1} - P_i)$ indicates that the applied forces acting on the i^{th} wedge exceed the forces resisting sliding along the base of the wedge. A positive value of the difference $(P_{i-1} - P_i)$ indicates that the applied forces acting on the i^{th} wedge are less than the forces resisting sliding along the base of that wedge. The governing equation for $(P_{i-1} - P_i)$ applies to the individual wedges. For the system of wedges to act as an integral failure mechanism, the safety factors of all wedges must be identical.

$$FS_1 = FS_2 = \dots = FS_{i-1} = FS_i = FS_{i+1} = \dots = FS_N$$

where N = the number of wedges in the failure mechanism.

The actual safety factor for sliding equilibrium is determined by satisfying overall horizontal equilibrium ($\Sigma F_H = 0$) for the entire system of wedges.

$$\sum_{i=1}^N (P_{i-1} - P_i) = 0 \quad \text{and} \quad P_0 \equiv 0 \quad P_N \equiv 0$$

Usually, an iterative solution process is used to determine the actual safety factor for sliding equilibrium. An example of a typical static loading condition analysis for a multiple-wedge system is presented in Example D2 of Appendix D. Note that if $\Sigma F_H < 0$, the factor of safety is less than the trial factor of safety, and if $\Sigma F_H > 0$, the factor of safety is greater than the trial factor of safety.

2. Critical Slip Angle for Driving-Side Wedge with Wall Friction on the Vertical Face

This section illustrates derivation of the critical slip angle for the limit equilibrium condition. Figure E-1 shows the forces acting on a driving-side wedge.

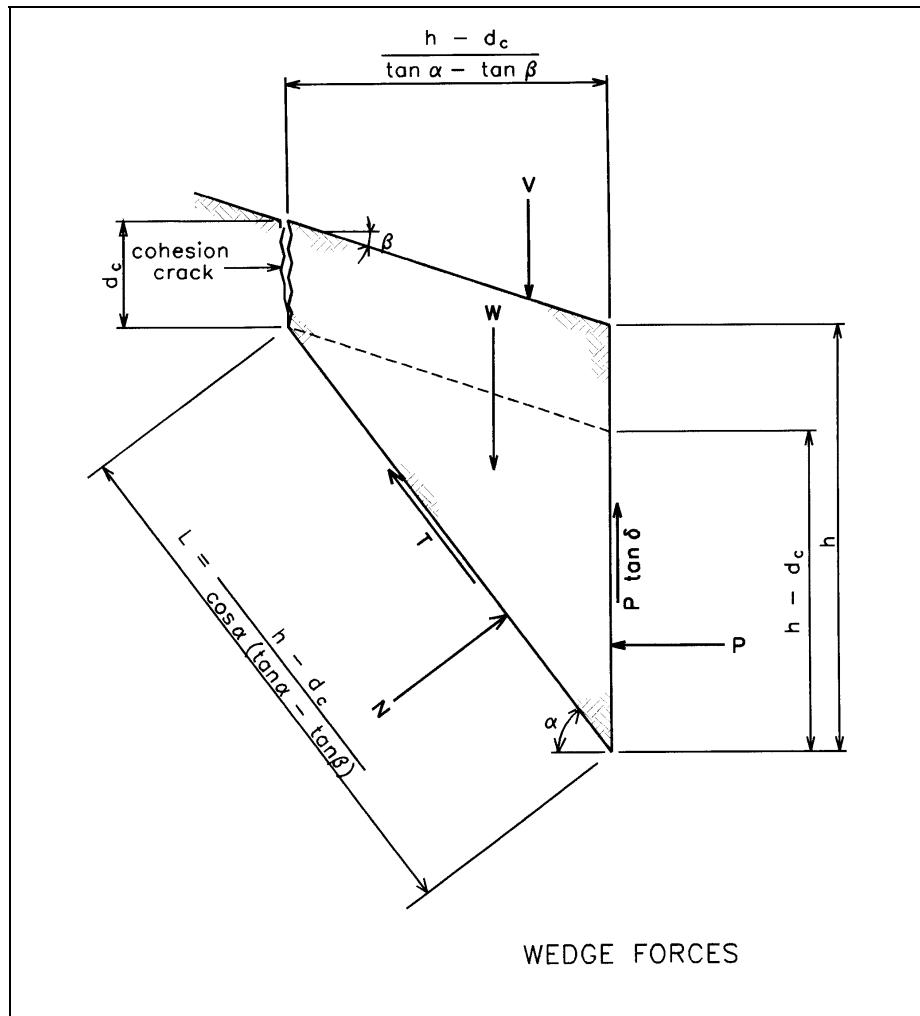


Figure E-1 Forces acting on wedge

For equilibrium to exist:

$$N = (W + V) \cos \alpha + P \sin \alpha - P \tan \delta \cos \alpha$$

$$T = (W + V) \sin \alpha - P \cos \alpha - P \tan \delta \sin \alpha$$

$$T \text{ must also} = N \tan \phi + cL$$

$$T = N \tan \phi + cL = (W + V) \tan \phi \cos \alpha + P \tan \phi \sin \alpha - P \tan \delta \tan \phi \cos \alpha + cL$$

Equating the two expressions for T , dividing them by $\cos \alpha$, and solving for P , the following is obtained:

$$P = \frac{(W + V)(\tan \alpha - \tan \phi) - \frac{cL}{\cos \alpha}}{(1 - \tan \delta \tan \phi) + (\tan \delta + \tan \phi) \tan \alpha}$$

where

W = weight of soil in wedge

V = strip surcharge

$$W = \frac{\gamma(h^2 - d_c^2)}{2(\tan \alpha - \tan \beta)}, \quad \gamma = \text{unit weight of soil}$$

$$L = \frac{h - d_c}{\cos \alpha (\tan \alpha - \tan \beta)}$$

substituting the above values for W and L into the equation for P :

$$P = \frac{\left[\frac{\gamma(h^2 - d_c^2)}{2(\tan \alpha - \tan \beta)} + V \right] (\tan \alpha - \tan \phi) - \frac{c(h - d_c)}{\cos^2 \alpha (\tan \alpha - \tan \beta)}}{(1 - \tan \delta \tan \phi) + (\tan \delta + \tan \phi) \tan \alpha}$$

Note that $t : \frac{1}{\cos^2 \alpha} = \sec^2 \alpha = 1 + \tan^2 \alpha$

divide both sides of this equation by:

$$\frac{\gamma(h^2 - d_c^2)}{2}, \quad \text{also substitute } 1 + \tan^2 \alpha \text{ for } \frac{1}{\cos^2 \alpha}$$

to obtain:

$$\frac{2P}{\gamma(h^2 - d_c^2)} = \frac{(\tan \alpha - \tan \phi) + \frac{2V}{\gamma(h^2 - d_c^2)} (\tan \alpha - \tan \beta)(\tan \alpha - \tan \phi) - \frac{2c(1 + \tan^2 \alpha)}{\gamma(h + d_c)}}{(\tan \alpha - \tan \beta)[1 - \tan \delta \tan \phi + (\tan \delta + \tan \phi) \tan \alpha]} = \frac{m}{n}$$

combining terms the above equation becomes:

$$\frac{m}{n} = \frac{(\tan \alpha - \tan \phi) + \frac{2V}{\gamma(h^2 - d_c^2)} [\tan^2 \alpha - (\tan \beta + \tan \phi) \tan \alpha + \tan \beta \tan \phi] - \frac{2c(1 + \tan^2 \alpha)}{\gamma(h + d_c)}}{(\tan \delta + \tan \phi) \tan^2 \alpha + [1 - \tan \delta \tan \phi - \tan \beta (\tan \delta + \tan \phi)] \tan \alpha - \tan \beta (1 - \tan \delta \tan \phi)}$$

The necessary condition for P to be either a maximum or a minimum is that the derivative of m/n , with respect to α , be equal to zero. The derivative of m/n is:

$$\frac{d(m/n)}{d\alpha} = \frac{n \left(\frac{dm}{d\alpha} \right) - m \left(\frac{dn}{d\alpha} \right)}{n^2} = 0$$

from this it can be seen that if both sides of the equation are multiplied by n^2 the maxima-minima condition becomes:

$$n \frac{dm}{d\alpha} - m \frac{dn}{d\alpha} = 0$$

After differentiating this equation, it will be found that all terms are multiplied by $\sec^2 \alpha$ ($d\tan\alpha/d\alpha$). For simplification, both sides of the equation will be divided (at the start) by $\sec^2\alpha$ to eliminate it.

Since all of the terms in the numerator (m) are over a common denominator, differentiation will be done separately for each numerator term; they will be combined at the end to furnish a complete solution. The derivative $dn/d\alpha$ is constant for all terms.

$$\frac{dn}{d\alpha} = 2(\tan\delta + \tan\phi)\tan\alpha + I - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)$$

The "W" term ($\tan\alpha - \tan\phi$):

$$\frac{dm}{d\alpha} = I$$

$$n \frac{dm}{d\alpha} = (\tan\delta + \tan\phi)\tan^2\alpha + (I - \tan\delta\tan\phi)\tan\alpha - \tan\beta(\tan\delta + \tan\phi)\tan\alpha - \tan\beta(I - \tan\delta\tan\phi)$$

$$\begin{aligned} m \frac{dn}{d\alpha} &= 2(\tan\delta + \tan\phi)\tan^2\alpha + (I - \tan\delta\tan\phi)\tan\alpha - \tan\beta(\tan\delta + \tan\phi)\tan\alpha \\ &\quad - 2\tan\phi(\tan\delta + \tan\phi)\tan\alpha - \tan\phi(I - \tan\delta\tan\phi) + \tan\beta\tan\phi(\tan\delta + \tan\phi) \end{aligned}$$

$$n \frac{dm}{d\alpha} - m \frac{dn}{d\alpha} = -(\tan\delta + \tan\phi)\tan^2\alpha + 2\tan\phi(\tan\delta + \tan\phi)\tan\alpha + \tan\phi - \tan\beta - (\tan\delta + \tan\beta)\tan^2\phi = 0$$

The "V" term:

$$\begin{aligned} \frac{dm}{d\alpha} &= \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] \tan\alpha - \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] (\tan\beta + \tan\phi) \\ n \frac{dm}{d\alpha} &= \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] (\tan\delta + \tan\phi)\tan^3\alpha + \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] [I - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\tan^2\alpha \\ &\quad - \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] \tan\beta(I - \tan\delta\tan\phi)\tan\alpha - \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] (\tan\beta + \tan\phi)(\tan\delta + \tan\phi)\tan^2\alpha \\ &\quad - \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] (\tan\beta + \tan\phi)[I - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\tan\alpha \\ &\quad + \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] \tan\beta(\tan\beta + \tan\phi)(I - \tan\delta\tan\phi) \end{aligned}$$

$$\begin{aligned}
 m \frac{dn}{d\alpha} &= \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] [(\tan\delta + \tan\phi)\tan^3\alpha - (\tan\delta + \tan\phi)(\tan\beta + \tan\phi)\tan^2\alpha + \tan\beta\tan\phi(\tan\delta + \tan\phi)\tan\alpha] \\
 &\quad + \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)] [\tan^2\alpha - (\tan\beta + \tan\phi)\tan\alpha] \\
 &\quad + \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)] \tan\beta\tan\phi = 0 \\
 n \frac{dm}{d\alpha} - m \frac{dn}{d\alpha} &= \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] (1 + \tan^2\phi)\tan^2\alpha - \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] \tan\beta(1 + \tan^2\phi)\tan\alpha \\
 &\quad + \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] \tan^2\beta(1 + \tan^2\phi) = 0
 \end{aligned}$$

The "c" term:

$$\begin{aligned}
 m &= -\frac{2c}{\gamma(h + d_c)} - \frac{2c\tan^2\alpha}{\gamma(h + d_c)}, \quad \frac{dm}{d\alpha} = -\frac{4c\tan\alpha}{\gamma(h + d_c)} \\
 \frac{dn}{d\alpha} &= 2(\tan\delta + \tan\phi)\tan\alpha + [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)] \\
 n \frac{dm}{d\alpha} &= -\left[\frac{4c}{\gamma(h + d_c)} \right] (\tan\delta + \tan\phi)\tan^3\alpha \\
 &\quad - \left[\frac{4c}{\gamma(h + d_c)} \right] [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)] \tan^2\alpha \\
 &\quad + \left[\frac{4c}{\gamma(h + d_c)} \right] [\tan\beta(1 - \tan\delta\tan\phi)] \tan\alpha \\
 m \frac{dn}{d\alpha} &= -\left[\frac{4c}{\gamma(h + d_c)} \right] (\tan\delta + \tan\phi)\tan^3\alpha \\
 &\quad - \left[\frac{2c}{\gamma(h + d_c)} \right] [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)] \tan^2\alpha \\
 &\quad - \left[\frac{4c}{\gamma(h + d_c)} \right] (\tan\delta + \tan\phi)\tan\alpha \\
 &\quad - \left[\frac{2c}{\gamma(h + d_c)} \right] [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]
 \end{aligned}$$

$$\begin{aligned}
 n \frac{dm}{d\alpha} - m \frac{dn}{d\alpha} = & - \left[\frac{2c}{(h+d_c)} \right] [1 - \tan \delta \tan \phi - \tan \beta (\tan \delta + \tan \phi)] \tan^2 \alpha \\
 & + \left[\frac{4c}{\gamma(h+d_c)} \right] [\tan \beta (1 - \tan \delta \tan \phi) + \tan \delta + \tan \phi] \tan \alpha \\
 & + \left[\frac{2c}{\gamma(h+d_c)} \right] [1 - \tan \delta \tan \phi - \tan \delta \tan \beta - \tan \beta \tan \phi] = 0
 \end{aligned}$$

Combining "W", "V", and "c" terms:

$$\begin{aligned}
 & -[\tan \phi + \tan \delta - \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] (1 + \tan^2 \phi) + \left[\frac{2c}{\gamma(h+d_c)} \right] (1 - \tan \delta \tan \phi - \tan \beta [\tan \delta + \tan \phi])] \tan^2 \alpha \\
 & + [2 \tan \phi (\tan \delta + \tan \phi) - \left[\frac{4V}{\gamma(h^2 - d_c^2)} \right] \tan \beta (1 + \tan^2 \phi) \\
 & + \left[\frac{4c}{\gamma(h+d_c)} \right] (\tan \beta + \tan \phi + \tan \delta [1 - \tan \beta \tan \phi])] \tan \alpha \\
 & + [\tan \phi - \tan \beta - (\tan \delta + \tan \beta) \tan^2 \phi + \left[\frac{2V}{\gamma(h^2 - d_c^2)} \right] \tan^2 \beta (1 + \tan^2 \phi) \\
 & + \left[\frac{2c}{\gamma(h+d_c)} \right] [1 - \tan \delta \tan \phi - \tan \beta (\tan \delta + \tan \phi)] = 0
 \end{aligned}$$

In order to make the above equation less cumbersome, let:

$$\begin{aligned}
 1 - \tan \delta \tan \phi - \tan \beta (\tan \delta + \tan \phi) &= r \\
 \tan \beta + \tan \phi + \tan \delta (1 - \tan \beta \tan \phi) &= s \\
 \tan \phi - \tan \beta - (\tan \delta + \tan \beta) \tan^2 \phi &= t
 \end{aligned}$$

Then the equation becomes:

$$\begin{aligned}
 & - \left[\tan \phi + \tan \delta - \left(\frac{2V}{\gamma(h^2 - d_c^2)} \right) (1 + \tan^2 \phi) + \left(\frac{2c}{\gamma(h+d_c)} \right) r \right] \tan^2 \alpha \\
 & + \left[2 \tan \phi (\tan \delta + \tan \phi) - \left(\frac{4V}{\gamma(h^2 - d_c^2)} \right) \tan \beta (1 + \tan^2 \phi) + \left(\frac{4c}{\gamma(h+d_c)} \right) s \right] \tan \alpha \\
 & + \left[t + \left(\frac{2V}{\gamma(h^2 - d_c^2)} \right) \tan^2 \beta (1 + \tan^2 \phi) + \left(\frac{2c}{\gamma(h+d_c)} \right) r \right] = 0
 \end{aligned}$$

Denoting the coefficient of $\tan^2 \alpha$ as $-A$, the coefficient of $\tan \alpha$ as AC_1 , and the constant as AC_2 we have the following quadratic equation:

$$-A \tan^2 \alpha + AC_1 \tan \alpha + AC_2 = 0$$

Dividing this by $-A$ gives:

$$\tan^2 \alpha - C_1 \tan \alpha - C_2 = 0$$

The solution for $\tan \alpha$ then is:

$$\tan \alpha = \frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2}$$

alternately

$$\alpha = \tan^{-1} \left(\frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2} \right)$$

where

$$A = \tan \phi + \tan \delta - \left(\frac{2V}{\gamma(h^2 - d_c^2)} \right) (1 + \tan^2 \phi) + \left(\frac{2c}{\gamma(h + d_c)} \right) r$$

$$C_1 = \frac{2 \tan \phi (\tan \delta + \tan \phi) - \left(\frac{4V}{\gamma(h^2 - d_c^2)} \right) \tan \beta (1 + \tan^2 \phi) + \left(\frac{4c}{\gamma(h + d_c)} \right) s}{A}$$

$$C_2 = \frac{t + \left(\frac{2V}{\gamma(h^2 - d_c^2)} \right) \tan^2 \beta (1 + \tan^2 \phi) + \left(\frac{2c}{\gamma(h + d_c)} \right) r}{A}$$